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Eshelby's tensor for embedded inclusions and the Elasto-Capillary phenomenon

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Abstract The elastic state of an embedded inclusion undergoing a stress-free transformation strain was the subject of John Douglas Eshelby's now classical paper in 1957. This paper, the subject of which is now widely known as "Eshelby's inclusion problem", is arguably one of the most cited papers in solid mechanics and several other branches of physical sciences. Applications have ranged from geophysics, quantum dots to composites. Over the past two decades, due to an interest in all things "small", attempts have been made to extend Eshelby's elastic analysis to the nanoscale by incorporating capillary or surface energy effects. In this note, we revisit a particular formulation that derives a very general expression for the elasto-capillary state of an embedded inclusion. This approach, that closely mimics that of Eshelby's original paper, appears to have the advantage that it can be readily used for inclusions of arbitrary shape (for numerical calculations) and provides a facile route for approximate solutions when closed-form expressions are not possible. Specifically, in the case of inclusions of constant curvature (sphere, cylinder) subject to some simplifications, closed-form expressions are obtained.

Keywords Elasto-capillary; surface energy; Eshelby's inclusion.

1. Introduction

Homogenization of composites, quantum dots, plasticity, among others are just some examples of the physical problems that have been tackled by Eshelby's solution to the elastic state of an embedded inclusion. Given the vast literature on the topic including several review papers and books, we avoid a detailed review of the literature and merely point to some relatively recent textbooks on this topic [Qu and Cherkaoui, 2006; Li and Wang, 2008; Cai and Nix, 2016]. In parallel, the capillary phenomenon has attracted attention from scientists from various disciplines

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and permeates disciplines ranging from catalysis to self-assembly, see, for example, a review by Müller and Saúl [2004].

The Eshelby's inclusion problem is the following: what is the elastic state of an embedded inclusion undergoing an inelastic transformation strain? This transformation strain may be due to thermal mismatch, phase transformation, plastic event, lattice mismatch among many other sources. The transformation strain, also known as eigenstrain (a term coined by Mura [1987]), is the strain state that will be achieved by the inclusion were it to be removed from the surrounding matrix body. However, due to the matrix constraint, the actual strain state is different. The fourth-order tensor that relates the eigenstrain to the actual strain state is known as "Eshelby's tensor". For certain shapes of inclusions, ellipsoids and the so-called E-inclusions [Liu *et al.*, 2007; Liu, 2008], this tensor reflects the fact that the strain state in the interior of such an inclusion is uniform (provided that we are under the realm of linearized elasticity and the eigenstrain is uniform). This fact greatly facilitates the solution to the problem where the inclusion and matrix region have differing elastic properties and has become the basis for much of the homogenization schemes for composites.

Eshelby's original work [Eshelby, 1957, 1959] did not include the capillary phenomenon and surface energy effects are expected to become significant at the nanoscale. Several works have extended Eshelby's idea to include surface energy including a few by the second author [Sharma *et al.*, 2003; Sharma and Ganti, 2004; Sharma and Wheeler, 2007]. In this short note, we revisit the formulation where a Green's function approach is used to derive a general expression for Eshelby's tensor. This approach, in our opinion, has the advantage that the expression itself is general and even if closed-form expression may not be possible in some context, it can be used as a starting point for an approximate solution. This paper follows much of the details outlined in the work by Sharma and Ganti [2004].

We would, in particular, like to single out a very thorough book chapter by Huang and Wang (in the book by Li and Gao [2013]) and other papers [Huang and Wang, 2006; Huang and Sun, 2007] which present several subtleties regarding surface energy effects (including a few that we gloss over in this present paper).

2. Formulation

The mechanical behavior of curved interfaces between solid phases incorporating surface energy, tension and stress has been well studied based on the mathematical framework developed by Gurtin and Murdoch [1975, 1978] and Gurtin *et al.* [1998]. Huang and Wang [2006] proposed a variational formulation for finite-deformation hyperelastic solids with the surface/interface energy effect:

$$\Pi(\mathbf{u}) = \int_{\partial\Omega} J_2\gamma(\mathbf{C}_S) + \int_D \rho_0\psi_0(\tilde{\mathbf{C}}) - \int_D \rho_0\mathbf{f}\cdot\mathbf{u} - \int_{\partial D_t} \bar{\mathbf{t}}_0\cdot\mathbf{u}, \tag{2.1}$$

where $\mathbf{u}$ is the displacement with the prescribed displacement $\bar{\mathbf{u}}_0$ on the boundary ∂D_u, $\rho_0\mathbf{f}$ is the body force in D, $\bar{\mathbf{t}}_0$ is the traction on the traction boundary $\partial D_t = \partial D \setminus \partial D_u$, J_2 is the ratio of the area element between the current and reference configurations, γ is the surface energy per unit area in the current configuration, and $\rho_0\psi_0$ is the hyperelastic potential. Moreover, $\mathbf{C}_{_S} = \mathbf{F}_{_S}^T\mathbf{F}_{_S}$ is the right Cauchy–Green tensor of the interface and $\mathbf{F}_{_S}$ is the surface deformation gradient that has the polar decomposition $\mathbf{F}_{_S} = \mathbf{R}_{_S}\mathbf{U}_{_S} = \mathbf{V}_{_S}\mathbf{R}_{_S}$. In contrast to $\mathbf{C}_{_S}$, $\tilde{\mathbf{C}}$ is defined as $(\mathbf{F}\mathbf{F}^*)^T(\mathbf{F}\mathbf{F}^*)$ that considers the "residual" elastic field induced by interface energy.

Vanishing of the first variation of the energy functional of Eq. (2.1) gives

$$\nabla \cdot \boldsymbol{\sigma}^B + \rho_0\mathbf{f} = \mathbf{0} \qquad \text{in } D, \tag{2.2}$$

$$\boldsymbol{\sigma}^B\mathbf{n} = \bar{\mathbf{t}}_0 \qquad \text{on } \partial D_t, \tag{2.3}$$

$$\mathbf{n} \cdot [\![\boldsymbol{\sigma}^B]\!]\mathbf{n} + \boldsymbol{\sigma}^S_{(in)} : \mathbf{b}_0 + \nabla_S \cdot [\mathbf{n} \cdot \boldsymbol{\sigma}^S_{(ou)}] = 0 \qquad \text{on } \partial\Omega, \tag{2.4}$$

$$\mathbf{P}^S[\![\boldsymbol{\sigma}^B]\!]\mathbf{n} + \nabla_S \cdot \boldsymbol{\sigma}^S_{(in)} + \mathbf{n} \cdot \boldsymbol{\sigma}^S_{(ou)}\mathbf{b}_0 = \mathbf{0} \qquad \text{on } \partial\Omega, \tag{2.5}$$

$$[\![\boldsymbol{\sigma}^S]\!]\mathbf{n} = \mathbf{0} \qquad \text{on } \partial\Omega_0 \subset \partial\Omega, \tag{2.6}$$

where the superscripts B and S indicate respectively bulk and surface, $\boldsymbol{\sigma}^B$ is the (nominal) bulk stress, $\boldsymbol{\sigma}^S$ is the (nominal) surface stress, $\boldsymbol{\sigma}^S_{(in)}$ is the "in-plane term" while $\boldsymbol{\sigma}^S_{(ou)}$ is the "out-plane term" of $\boldsymbol{\sigma}^S$, $\mathbf{n}$ is the outward unit normal to the corresponding surfaces, and the jump $[\![\cdot]\!]$ denotes the difference $(\cdot)^+ - (\cdot)^-$ across the surfaces. Detailed derivations of Eqs. (2.2)–(2.6) can be found in the work of Huang and Wang [2006] or in the book chapter by Huang and Wang [Li and Gao, 2013].

Note that $\mathbf{P}^S$ in Eq. (2.5) is the surface projection tensor that transforms vectors into the tangent space and vice versa is defined as [Gurtin and Murdoch, 1975, 1978; Gurtin *et al.*, 1998]

$$\mathbf{P}^S = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}, \tag{2.7}$$

where $\mathbf{I}$ is the unit tensor. The surface divergence $(\nabla_S \cdot)$ and the surface gradient (∇_S) are defined by

$$\nabla_S \cdot \mathbf{v} = \mathrm{Tr}(\nabla_S\mathbf{v}) \tag{2.8}$$

and

$$\nabla_S\mathbf{v} = (\nabla\mathbf{v})\mathbf{P}^S, \tag{2.9}$$

where $\mathbf{v}(\mathbf{x})$ is a smooth field with $\mathbf{v}$ vector valued, "Tr" denotes the trace of a tensor, and ∇ is the well-known gradient in three dimensions. In particular, the surface divergence of the surface projection tensor $\mathbf{P}^S$ is given by [Gurtin and Murdoch, 1975; Gurtin *et al.*, 1998]

$$\nabla_S \cdot \mathbf{P}^S = -(\nabla_S \cdot \mathbf{n})\mathbf{n} = \mathrm{Tr}(\mathbb{L})\mathbf{n} = 2\kappa\mathbf{n}, \tag{2.10}$$

where $\mathbb{L} = -\nabla_S\mathbf{n}$ is the curvature tensor and κ is the mean curvature.

From the hyperelastic potential $\rho_0\psi_0(\tilde{\mathbf{C}})$ and the surface energy $J_2\gamma(\mathbf{C}_S)$ in Eq. (2.1), the (nominal) bulk stress $\boldsymbol{\sigma}^B$ and the (nominal) surface stress $\boldsymbol{\sigma}^S$ in Eqs. (2.2)–(2.6), can be written, like Huang and Wang [2006] and Li and Gao [2013], as

$$\boldsymbol{\sigma}^B = 2\rho_0\mathbf{F}\mathbf{F}^*\frac{\partial\psi_0}{\partial\tilde{\mathbf{C}}}\mathbf{F}^{*T} \tag{2.11}$$

and

$$\boldsymbol{\sigma}^S = 2\mathbf{F}_S\frac{\partial(J_2\gamma)}{\partial\mathbf{C}_S}. \tag{2.12}$$

Linearization of $\boldsymbol{\sigma}^B$ in Eq. (2.11) at $\mathbf{F} = \mathbf{I}$ gives

$$\boldsymbol{\sigma}^B = \boldsymbol{\sigma}^{*B} + \mathbb{C} : \boldsymbol{\varepsilon}, \tag{2.13}$$

where $\boldsymbol{\sigma}^{*B} = \boldsymbol{\sigma}^B|_{\mathbf{F}=\mathbf{I}}$ is the residual stress, $\mathbb{C} = \rho_0\frac{\partial^2\psi_0}{\partial\mathbf{E}^2}|_{\mathbf{E}=\mathbf{0}}$ is the elastic stiffness tensor, and the infinitesimal strain tensor is defined by

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T). \tag{2.14}$$

Neglecting the residual stress and considering homogeneously linear elastic isotropic materials, Eq. (2.13) can be further written as

$$\boldsymbol{\sigma}^B = \mathbb{C} : \boldsymbol{\varepsilon} = \lambda(\mathrm{Tr}\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}, \tag{2.15}$$

where λ and μ are the Lamé constants for the isotropic bulk material.

Similarly, the linearization of $\boldsymbol{\sigma}^S$ in Eq. (2.12) at $\mathbf{F} = \mathbf{I}$ gives

$$\boldsymbol{\sigma}^S = \boldsymbol{\sigma}^{*S} + \mathbb{C}_S : \boldsymbol{\varepsilon}^S, \tag{2.16}$$

where $\boldsymbol{\sigma}^{*S} = \tau_o\mathbf{P}^S$ and τ_o is the deformation independent surface/interfacial tension, and $\mathbb{C}_S$ is the elastic stiffness tensor of the surface, and

$$\boldsymbol{\varepsilon}^S = \mathbf{P}^S\boldsymbol{\varepsilon}\mathbf{P}^S. \tag{2.17}$$

For infinitesimal deformation, the relation between the interface/surface stress tensor, $\boldsymbol{\sigma}^S$, and the interface/surface strain tensor, $\boldsymbol{\varepsilon}^S$, can be written by Gurtin and Murdoch [1975, 1978] as

$$\boldsymbol{\sigma}^S = [\tau_o + (\lambda^S + \tau_o)\mathrm{Tr}(\boldsymbol{\varepsilon}^S)]\mathbf{P}^{\mathbf{s}} + 2(\mu^S - \tau_o)\boldsymbol{\varepsilon}^S + \tau_o\nabla_S\mathbf{u}, \tag{2.18}$$

where τ_o is the residual surface tension (under unstrained conditions), λ^S and μ^S are the surface Lamé constants for isotropic interfaces or surfaces. Without body force, the equilibrium and isotropic constitutive equations in the bulk and on the surface are written as usual:

$$\nabla\cdot\boldsymbol{\sigma}^{\mathbf{B}} = \mathbf{0} \quad \text{in } D\setminus\partial\Omega \tag{2.19}$$

and

$$[\![\boldsymbol{\sigma}^{\mathbf{B}}\mathbf{n}]\!] + \nabla_S\cdot\boldsymbol{\sigma}^S = \mathbf{0} \quad \text{on } \partial\Omega. \tag{2.20}$$

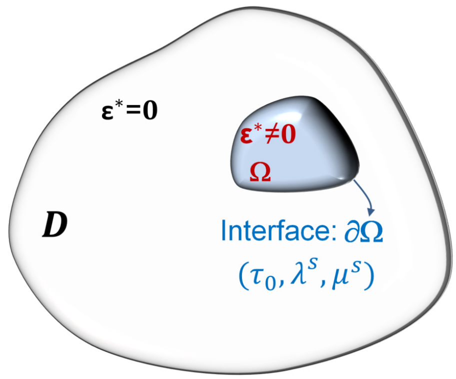

Fig. 1. Schematic of the problem.

Based now on the framework of the deformation of curved interfaces incorporating the surface/interface energy effect described earlier, we revisit the classical problem of Eshelby's inclusion [Eshelby, 1957, 1959]. An inclusion is defined as a subdomain Ω in a domain D, where eigenstrain $\boldsymbol{\varepsilon}^\star(\mathbf{x})$ is given in Ω and is zero in $D \setminus \Omega$ (shown schematically in Fig. 1). The material properties, such as the stiffness tensor $\mathbb{C}$, in Ω and in $D \setminus \Omega$ are the same, namely

$$\begin{cases} \boldsymbol{\varepsilon}^\star(\mathbf{x}) \neq \mathbf{0} & \text{in } \Omega, \\ \boldsymbol{\varepsilon}^\star(\mathbf{x}) = \mathbf{0} & \text{in } D \setminus \Omega. \end{cases} \tag{2.21}$$

The constitutive law, Eq. (2.15), becomes

$$\boldsymbol{\sigma}^B = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^\star(\mathbf{x})). \tag{2.22}$$

In particular, if the eigenstrain $\boldsymbol{\varepsilon}^\star(\mathbf{x})$ is uniform and equal to $\boldsymbol{\varepsilon}^*$ in Ω, a more concise expression of the eigenstrain admits the form

$$\boldsymbol{\varepsilon}^\star(\mathbf{x}) = H[z(\mathbf{x})]\boldsymbol{\varepsilon}^* \quad \text{in } D, \tag{2.23}$$

where H is the Heaviside function, and

$$\begin{cases} z(\mathbf{x}) > 0 & \text{in } \Omega, \\ z(\mathbf{x}) < 0 & \text{in } D \setminus \Omega. \end{cases} \tag{2.24}$$

Introducing capillary effects into the picture, the interface conditions on the displacement and the interfacial traction across the interface $\partial\Omega$ are

$$\begin{cases} [\![\boldsymbol{u}]\!] = \mathbf{0} \\ [\![\boldsymbol{\sigma}^{\mathbf{B}}\mathbf{n}]\!] = -\nabla_S \cdot \boldsymbol{\sigma}^S \end{cases} \quad \text{on } \partial\Omega. \tag{2.25}$$

Note that the displacement and the interfacial traction across the interface must be continuous for perfect interfaces in the classical inclusion problem.

Consider the Dirac delta function δ and take the constitutive law in Eq. (2.22) and the interface condition in Eq. (2.25). The equation of equilibrium, Eq. (2.19), is given by [Sharma and Ganti, 2004]

$$\nabla \cdot \boldsymbol{\sigma}^{\mathbf{B}} = \nabla \cdot (\mathbb{C} : \boldsymbol{\varepsilon}) \underbrace{- \nabla \cdot [\mathbb{C} : \boldsymbol{\varepsilon}^{\star}(\mathbf{x})]}_{\mathbf{f}^{I}} \underbrace{- \delta[\bar{z}(\mathbf{x})] [\![\boldsymbol{\sigma}^{\mathbf{B}} \mathbf{n}]\!]}_{\mathbf{f}^{S}} = \mathbf{0}. \tag{2.26}$$

It can be readily seen that both the first underlined term (from eigenstrain) and the second underlined term (from surface effects) appear as body forces $\mathbf{f}^{I}$ and $\mathbf{f}^{S}$. The interface is defined by $\bar{z}(\mathbf{x}) = 0$. Note that in classical inclusion problem the last underlined expression in Eq. (2.26) is typically omitted since the jump in the normal traction on perfect interfaces is zero.

With Eq. (2.20) or Eq. $(2.25)_2$, Eq. (2.26) can be rewritten as

$$\nabla \cdot \boldsymbol{\sigma}^{\mathbf{B}} = \nabla \cdot (\mathbb{C} : \boldsymbol{\varepsilon}) \underbrace{- \nabla \cdot \{\mathbb{C} : \boldsymbol{\varepsilon}^{\star}(\mathbf{x})\} + \delta[\bar{z}(\mathbf{x})] \nabla_{S} \cdot \boldsymbol{\sigma}^{S}}_{\mathbf{f}^{\mathrm{total}} = \mathbf{f}^{I} + \mathbf{f}^{S}} = \mathbf{0}. \tag{2.27}$$

Consider the symmetry of $\mathbb{C}$ and the infinitesimal strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^{T}) = \mathrm{sym}(\nabla \mathbf{u})$. Here $\mathrm{sym}(\cdot)$ denotes the symmetric part of a tensor, such that $\mathrm{sym}(\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{T})$. Using the underlined term in Eq. (2.27) as representing a total body force $\mathbf{f}^{\mathrm{total}}$ in conjunction with the elastic Green's function, we can write the displacement field and the strain tensor due to both the eigenstrain and the surface effect as [Sharma and Ganti, 2004]

$$\begin{aligned} \mathbf{u}(\mathbf{x}) = & \int_{D} (\mathbb{C} : \boldsymbol{\varepsilon}^{\star}) : (\nabla_{x} \otimes \mathbf{G}^{T}(\mathbf{y} - \mathbf{x}))\, dV(\mathbf{y}) \\ & + \int_{\partial\Omega} \mathbf{G}^{T}(\mathbf{y} - \mathbf{x}) \cdot [\nabla_{S} \cdot \boldsymbol{\sigma}^{S}(\mathbf{y})]\, dS(\mathbf{y}) \end{aligned} \tag{2.28}$$

and

$$\boxed{\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{S} : \boldsymbol{\varepsilon}^{\star} + \mathrm{sym}\left\{ \nabla_{x} \otimes \int_{\partial\Omega} \mathbf{G}^{T}(\mathbf{y} - \mathbf{x}) \cdot [\nabla_{S} \cdot \boldsymbol{\sigma}^{S}(\mathbf{y})]\, dS(\mathbf{y}) \right\}.} \tag{2.29}$$

Eshelby's interior tensor $\mathbf{S}$ in Eq. (2.29) is defined as

$$\mathbf{S} = \mathbb{P}^{\Omega} : \mathbb{C}, \tag{2.30}$$

where $\mathbb{P}^{\Omega}$ is a fourth-order tensor with the components

$$P_{ijkl}^{\Omega}(\mathbf{x}) = \int_{\Omega} \Sigma_{ijkl}(\mathbf{x} - \mathbf{y})\, dV(\mathbf{y}) \tag{2.31}$$

with

$$\begin{aligned} & \Sigma_{ijkl}(\mathbf{x} - \mathbf{y}) \\ & \quad = \frac{1}{4}\left[\frac{\partial^2 G_{ki}(\mathbf{x} - \mathbf{y})}{\partial x_j \partial y_l} + \frac{\partial^2 G_{kj}(\mathbf{x} - \mathbf{y})}{\partial x_i \partial y_l} + \frac{\partial^2 G_{li}(\mathbf{x} - \mathbf{y})}{\partial x_j \partial y_k} + \frac{\partial^2 G_{lj}(\mathbf{x} - \mathbf{y})}{\partial x_i \partial y_k} \right]. \end{aligned} \tag{2.32}$$

Here the reciprocal theorem, $G_{ij}(\mathbf{x}-\mathbf{y}) = G_{ji}(\mathbf{y}-\mathbf{x})$, is used and the Green's function for isotropic material is given by [Qu and Cherkaoui, 2006; Li and Wang, 2008]:

$$G_{ij}(\mathbf{x}-\mathbf{y}) = \frac{1}{16\pi\mu(1-\nu)|\mathbf{x}-\mathbf{y}|}\left\{(3-4\nu)\delta_{ij} + \frac{(x_i-y_i)(x_j-y_j)}{|\mathbf{x}-\mathbf{y}|^2}\right\}. \tag{2.33}$$

Further simplification of Eq. (2.29) does not appear feasible without additional assumptions regarding the shape of inclusion. Note that Eq. (2.29) implicitly gives the modified Eshelby tensor for inclusions incorporating the surface effects. This modified relation is implicit since the surface stress depends on the surface strain, which in turn is the projection of the conventional strain $\boldsymbol{\varepsilon}$ on the tangent plane of the inclusion–matrix interface. In the next section, using Eq. (2.29), we will derive explicit expressions for spherical and cylindrical inclusions. For now, however, it is worth noting some general features of the new Eshelby inclusion tensor.

Recall the relation $\boldsymbol{\varepsilon}^S = \mathbf{P}^S\boldsymbol{\varepsilon}\mathbf{P}^S$ and the surface constitutive law Eq. (2.18). The surface divergence of surface stress tensor in the modified Eshelby tensor Eq. (2.29) can only be uniform if the bulk strain $\boldsymbol{\varepsilon}$ and the projection tensor $\mathbf{P}^S$ are uniform over the inclusion surface. This is because the surface divergence of the projection tensor is given by $\nabla_S \cdot \mathbf{P}^S = 2\kappa\mathbf{n}$ in Eq. (2.10).

It is known that the curvature is non-uniform and varies depending upon the location at the surface of a general ellipsoid. Only for the two special cases of spherical and cylindrical inclusions admit a uniform mean curvature. Sharma and Ganti [2004] gave the following important proposition:

Proposition: "*Eshelby's original conjecture that only inclusions of the ellipsoid family admit uniform elastic state under uniform eigenstrains must be modified in the context of coupled surface/interface-bulk elasticity. Only inclusions that are of a constant curvature admit a uniform elastic state, thus restricting this remarkable property to spherical and cylindrical inclusions.*"

3. Spherical and Cylindrical Inclusions

Note now that further simplification of the modified Eshelby tensor in Eq. (2.29) can be made by regarding the shape of inclusion with a constant curvature. The new Eshelby tensor will be size-dependent by regarding the surface effects of inclusions.

Consider an inclusion with constant mean curvature κ. The surface divergence of the surface stress is remarkably reduced in the modified Eshelby tensor in Eq. (2.29), such that

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{S} : \boldsymbol{\varepsilon}^\star + \mathrm{sym}\left\{\nabla_x \otimes \int_{\partial\Omega} \mathbf{G}^T(\mathbf{y}-\mathbf{x}) \cdot (2\kappa s\mathbf{n})\, dS(\mathbf{y})\right\}$$

$$= \mathbf{S} : \boldsymbol{\varepsilon}^{\star} + 2\kappa s \left(\mathrm{sym} \left\{ \nabla_x \otimes \int_{\Omega} \nabla_y \cdot \mathbf{G}(\mathbf{y}-\mathbf{x})\ dV(\mathbf{y}) \right\} \right)$$
$$= \mathbf{S} : \boldsymbol{\varepsilon}^{\star} - 2\kappa s \mathbb{C}^{-1} : \left(\mathrm{sym} \left\{ \nabla_x \otimes \mathbb{C} : \int_{\Omega} \nabla_x \otimes \mathbf{G}(\mathbf{y}-\mathbf{x})\ dV(\mathbf{y}) \right\} : \mathbf{I} \right). \tag{3.1}$$

Here, we use the divergence theorem and some basic tensor algebra as well as the rule of the Green's function, $\nabla_x \otimes \mathbf{G}(\mathbf{y}-\mathbf{x}) = -\nabla_y \otimes \mathbf{G}(\mathbf{y}-\mathbf{x})$. Additionally, the surface constitutive law is used and the scalar "s" is defined from the relation $\boldsymbol{\sigma}^S = s\mathbf{P}^S$ [Sharma and Ganti, 2004], such that

$$s = \tau_o + (\lambda^S + \mu^S)\mathrm{Tr}(\mathbf{P}^S \boldsymbol{\varepsilon} \mathbf{P}^S). \tag{3.2}$$

Recall the form of the classical Eshelby tensor, such as Eqs. (2.30)–(2.32). The term enclosed in the curly brackets in Eq. (3.1) is exactly the classical Eshelby tensor, and then Eq. (3.1) can be cast into a more attractive form:

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{S} : \boldsymbol{\varepsilon}^{\star} - 2\kappa s \mathbb{C}^{-1} : (\mathbf{S} : \mathbf{I}). \tag{3.3}$$

Some basic tensor algebra are introduced here for convenience [Qu and Cherkaoui, 2006]. A fourth-order isotropic tensor $\mathbb{A}$ can be written as $\mathbb{A} = 3a\mathbb{I}^h + 2b\mathbb{I}^d$, where a and b are scalars, and the two fourth-order tensors are $I^h_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}$ and $I^d_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl})$. Here δ_{ij} is the Kronecker delta. The inverse is $\mathbb{A}^{-1} = \frac{1}{3a}\mathbb{I}^h + \frac{1}{2b}\mathbb{I}^d$. Introduce a second-order symmetric tensor $\boldsymbol{\epsilon}$. It can be easily shown that $\mathbb{A} : \boldsymbol{\epsilon} = 3a\epsilon_m\mathbf{I} + 2b\boldsymbol{\epsilon}'$, where $\epsilon_m\mathbf{I}$ and $\boldsymbol{\epsilon}'$ are the spherical part and the deviatoric part of $\boldsymbol{\epsilon}$, such that $\epsilon_m = \frac{1}{3}\mathrm{Tr}\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}' = \boldsymbol{\epsilon} - \epsilon_m\mathbf{I}$. In particular, $\mathbb{A} : \mathbf{I} = 3a\mathbf{I}$, where $\mathbf{I}$ is the second-order identity tensor.

It is known that the fourth-order stiffness tensor for isotropic materials can be written as $\mathbb{C} = 3K\mathbb{I}^h + 2\mu\mathbb{I}^d$ and the corresponding compliance tensor is given by $\mathbb{C}^{-1} = \frac{1}{3K}\mathbb{I}^h + \frac{1}{2\mu}\mathbb{I}^d$, where K and μ are the bulk and shear moduli. Moreover, the Eshelby tensor $\mathbf{S}$ for a spherical inclusion in an isotropic material is a fourth-order isotropic tensor:

$$\mathbf{S} = \frac{3K}{3K+4\mu}\mathbb{I}^h + \frac{6(K+2\mu)}{5(3K+4\mu)}\mathbb{I}^d. \tag{3.4}$$

Using Eq. (3.2) and the stiffness tensor as well as the property of the fourth-order isotropic tensor, Eq. (3.3) for spherical inclusions can be written as

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{S} : \boldsymbol{\varepsilon}^{\star} - \frac{K^S}{3KR_o}\mathrm{Tr}(\mathbf{P}^S \boldsymbol{\varepsilon} \mathbf{P}^S)(\mathbf{S} : \mathbf{I}) - \frac{2\tau_o}{3KR_o}(\mathbf{S} : \mathbf{I}), \tag{3.5}$$

where K^S is defined by us to be the surface elastic modulus that is equal to $2(\lambda^S + \mu^S)$ and $K = \lambda + 2\mu/3$ is the bulk modulus. We also have used the fact that the curvature of a spherical inclusion with radius R_o is $\kappa = 1/R_o$.

Similarly, for an infinite circular cylindrical inclusion in plane strain, Eq. (3.3) can be written as [Sharma and Ganti, 2004]

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{S} : \boldsymbol{\varepsilon}^{\star} - \frac{K'^{S}}{3K'R_o}\mathrm{Tr}(\mathbf{P}^S \boldsymbol{\varepsilon} \mathbf{P}^S)(\mathbf{S} : \mathbf{I}) - \frac{\tau_o}{3K'R_o}(\mathbf{S} : \mathbf{I}), \tag{3.6}$$

where $K'^S = \lambda^S + 2\mu^S$ is the plane-strain surface elastic modulus and $K' = 2(\lambda + \mu)/3$.

In the following, we will show the components of the strain tensors in Eqs. (3.5) and (3.6) incorporating the surface effects. Note that Eshelby's interior tensor $\mathbf{S}$ is used in the solution of interior strains while for exterior strains the corresponding exterior version $\mathbf{D}$ is required [Eshelby, 1959; Mura, 1987; Li and Wang, 2008]. Here we just show the detailed solution of the interior strain of spherical inclusions. For detailed solutions of the exterior strain of spherical inclusions and the strain of cylindrical inclusions, one can refer to the work by Sharma and Ganti [2004].

For a spherical inclusion with a constant eigenstrain $\boldsymbol{\varepsilon}^{\star}(\mathbf{r}) = \varepsilon^{*}\mathbf{I}$, $|\mathbf{r}| < R_o$, in spherical polar coordinates (r, θ, ϕ), the trace of the surface strain tensor is $\mathrm{Tr}(\mathbf{P}^S \boldsymbol{\varepsilon} \mathbf{P}^S) = \varepsilon_{\theta\theta} + \varepsilon_{\phi\phi}$ and then Eq. (3.5) becomes

$$\boldsymbol{\varepsilon}(\mathbf{r}) = \left\{ \varepsilon^{*} - \frac{K^S(\varepsilon_{\theta\theta} + \varepsilon_{\phi\phi})}{3KR_o} - \frac{2\tau_o}{3KR_o} \right\} (\mathbf{S} : \mathbf{I}), \quad \text{for } |\mathbf{r}| < R_o, \tag{3.7}$$

which, by using the Eshelby tensor $\mathbf{S}$ in Eq. (3.4) and the property of fourth-order isotropic tensors, can be further written as

$$\boldsymbol{\varepsilon}(\mathbf{r}) = \frac{1}{3K + 4\mu} \left\{ 3K\varepsilon^{*} - \frac{K^S(\varepsilon_{\theta\theta} + \varepsilon_{\phi\phi})}{R_o} - \frac{2\tau_o}{R_o} \right\} \mathbf{I}, \quad \text{for } |\mathbf{r}| < R_o. \tag{3.8}$$

Since the second-order identity tensor in spherical polar coordinates is $\mathbf{I} = \mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi$, the components of $\boldsymbol{\varepsilon}(\mathbf{r})$ in Eq. (3.8) can be easily obtained as

$$\begin{cases} \varepsilon_{rr}(\mathbf{r}) = \varepsilon_{\theta\theta}(\mathbf{r}) = \varepsilon_{\phi\phi}(\mathbf{r}) = \dfrac{3K\varepsilon^{*} - 2\tau_o/R_o}{3K + 4\mu + 2K^S/R_o}, \\ \varepsilon_{r\theta}(\mathbf{r}) = \varepsilon_{r\phi}(\mathbf{r}) = \varepsilon_{\theta\phi}(\mathbf{r}) = 0, \end{cases} \quad \text{for } |\mathbf{r}| < R_o. \tag{3.9}$$

Similarly, by using Eshelby's exterior tensor $\mathbf{D}$, Sharma and Ganti [2004] gave the strain field outside of the spherical inclusion

$$\begin{cases} \varepsilon_{rr}(\mathbf{r}) = \dfrac{3K\varepsilon^{*} - 2\tau_o/R_o}{3K + 4\mu + 2K^S/R_o}\dfrac{R_o^3}{r^3}, \ \varepsilon_{\theta\theta}(\mathbf{r}) = \varepsilon_{\phi\phi}(\mathbf{r}) = -2\varepsilon_{rr}(\mathbf{r}), \\ \varepsilon_{r\theta}(\mathbf{r}) = \varepsilon_{r\phi}(\mathbf{r}) = \varepsilon_{\theta\phi}(\mathbf{r}) = 0, \end{cases} \quad \text{for } |\mathbf{r}| > R_o. \tag{3.10}$$

In contrast to the spherical inclusion, the strain field of cylindrical inclusions under plane strain in cylindrical polar coordinates (ρ, ϕ, z) is given by

$$\begin{cases} \varepsilon_{\rho\rho}(\mathbf{r}) = \varepsilon_{\phi\phi}(\mathbf{r}) = \dfrac{3K'\varepsilon^* - \tau_o/R_o}{3K' + 2\mu + K'^S/R_o}, \quad \text{for } \rho < R_o, \\ \varepsilon_{\rho\rho}(\mathbf{r}) = -\varepsilon_{\phi\phi}(\mathbf{r}) = \dfrac{3K'\varepsilon^* - \tau_o/R_o}{3K' + 2\mu + K'^S/R_o}\dfrac{R_o^2}{\rho^2}, \quad \text{for } \rho > R_o, \\ \varepsilon_{\rho\phi}(\mathbf{r}) = \varepsilon_{zz}(\mathbf{r}) = 0. \end{cases} \tag{3.11}$$

Equations (3.9)–(3.11) are exceptionally simple but clearly illustrate the size-dependent elastic state. The surface/interface tension is a residual strain-type term which, for example, should not impact the effective properties of composite unless non-linear effects are carefully accounted for and linearization is properly performed. We refer the reader to the chapter and papers by Huang and co-workers on this matter [Li and Gao, 2013; Huang and Wang, 2006; Huang and Sun, 2007]. The effect of surface elasticity appears through K^S, leading to a size-dependent change in overall hydrostatic properties of a composite. By making the radius of the inclusion large we can trivially retrieve the well-known classical solution. Using an assumed displacement type method, Cahn and Lärche [1982] (only taking into account surface tension) presented exactly the expression in Eq. (3.9) with the surface elasticity effect (K^S) set to zero. The application, including the size-dependent stress concentration at a spherical void and the size-dependent overall properties of composites, among others, can be found in the work by Sharma and Ganti [2004].

4. Concluding Remarks

Our expression Eq. (2.29) can be used as the starting point for approximations to problems which do not yield closed-form expressions. Exact solution is only currently known for the radially symmetric case, i.e., spherical and cylindrical inclusions with radial and uniform eigenstrains. Approximate solutions using multipole expansion have been derived by Duan *et al.* [2005]. Our approach may also be used to find approximate solutions, for example, for ellipsoidal or polyhedral shape inclusions.

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